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Harnesses, Lévy bridges and *Monsieur Jourdain*

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Abstract

Relations between harnesses and initial enlargements of the filtration of a Lévy process with its positions at fixed times are investigated.

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1. Introduction

In order to model long-range misorientation within crystalline structure of metals, Hammersley [9] introduced various notions of processes which enjoy particular conditional expectation properties. Among these, harnesses (see the following definition) are of particular interest.

Definition 1. Let $(H_t; t \geqslant 0)$ be a measurable process such that for all t, $\mathbb{E}[|H_t|] < \infty$, and define for all $t < T$

$$\mathscr{H}_{t,T} := \sigma\{H_s; s \leqslant t; H_u; u \geqslant T\}.$$

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H is said to be a harness if, for all $a < b < c < d$

$$\mathbb{E}\left[\frac{H_c - H_b}{c - b}\middle|\mathscr{H}_{a,d}\right] = \frac{H_d - H_a}{d - a}. \tag{1}$$

One may also define the notion of a $\mathbb{F} = (\mathscr{F}_{t,T})_{t<T}$-harness as soon as $\mathscr{H}_{t,T} \subset \mathscr{F}_{t,T}$, with some obvious hypothesis on a "past–future" filtration $\mathbb{F}$; this notion may be just as useful as the notion of Brownian motion with respect to a filtration. Identity (1) may be reformulated as follows: H is a harness if and only if for all $s < t < u$

$$\mathbb{E}[H_t|\mathscr{H}_{s,u}] = \frac{t-s}{u-s} H_u + \frac{u-t}{u-s} H_s. \tag{2}$$

Such a formulation justifies that harnesses are sometimes called affine processes (see [2, Chapter 6]).

We note that Williams [23,24] proved the following striking result: the only square integrable continuous harnesses are Brownian motions with drifts, more precisely: $\sigma B_t + \mu t,\ \ t \geqslant 0$, where σ and μ are measurable with respect to the σ-field $\bigcap_{0<t<T<\infty} \mathscr{F}_{t,T}$. This latter result shows how rigid the property of being a continuous harness is and may help understand why there has been so few studies of harnesses with continuous time during the past 20 years. On the other hand, some multi-parameter versions and discussions appeared, imitating Williams' arguments (see [4,25–27]).

Glancing through the literature, it seems that no systematic study of discontinuous harnesses has yet appeared. Our reference to Monsieur Jourdain (a character of Molière (1622–1673) [17]) in the title alludes to this point; Monsieur Jourdain discovers that he is practicing prose without being aware of it; analogously the following theorem shows that a number of authors have been dealing with harnesses:

Theorem 2. (i) (*Jacod–Protter* [12]; *see also* [20, *Chapter VI*]) *Let* $(\xi_t; t \geqslant 0)$ *be an integrable Lévy process* (*that is*: $\forall t, \mathbb{E}[|\xi_t|] < \infty$) *and define*

$$\mathscr{F}_{t,T} = \sigma\{\xi_s; s \leqslant t; \xi_u; u \geqslant T\}.$$

Then for any given $T > 0$, *there is the decomposition formula*

$$\xi_t = M_t^{(T)} + \int_0^t \mathrm{d}s\, \frac{\xi_T - \xi_s}{T - s}, \tag{3}$$

where $(M_t^{(T)}; t \leqslant T)$ *is a* $(\mathscr{F}_{t,T}; t \leqslant T)$-*martingale*.

(ii) *In a general framework, an integrable process* $(H_t; t \geqslant 0)$ *is a* $(\mathscr{F}_{t,T})$-*harness if and only if, for every* $T > 0$, *there exists* $(M_t^{(T)})_{t<T}$ *a* $(\mathscr{F}_{t,T}; t < T)$-*martingale such that*

$$\forall t < T, \quad H_t = M_t^{(T)} + \int_0^t \mathrm{d}s \frac{H_T - H_s}{T - s}. \tag{4}$$

For further results along these lines, see Exercise 6.19 in [2] which provides a few references about harnesses. In the particular case of a Brownian motion ξ, formula

(3) may be attributed to Itô [11] but was already sketched by Lévy [16,17]. See also [13,18]. Our motivation for writing this note is that harnesses—through formula (3)—have become quite topical; indeed some recent works [3,14] develop the financial models of markets with well-informed agents (also called insiders), where the formula (4) plays a key-role. Some other papers [6,7,10,15,22] also deal with some notions of harness derived directly from the pioneering work of Hammersley, but are apparently far from the preceding discussion.

This note is organized as follows:

- In Section 2, we prove part (ii) of the theorem.
- Section 3 is devoted to an alternative proof of the decomposition formula (3) of Jacod–Protter [12], thanks to the (partial) absolute continuity results between the law of a Lévy process and its bridges.
- In Section 4, we develop the more general notion of a past–future martingale and provide some examples.

2. Relations between Lévy bridges and harnesses

(2.1). Let $(B_t; t \geqslant 0)$ be a one-dimensional Brownian motion; it is well known that a realization of the Brownian bridge over the time interval $[0, T]$, starting at x and ending at y, is

$$\left\{x + \left(B_t - \frac{t}{T} B_T\right) + \frac{t}{T} y; t \leqslant T\right\}. \tag{5}$$

Moreover, the semimartingale decomposition of this bridge is also well known; it is the solution of the SDE

$$X_t = x + \beta_t + \int_0^t \mathrm{d}s \, \frac{y - X_s}{T - s}; \; t \leqslant T, \tag{6}$$

where $(\beta_t; t \leqslant T)$ is a standard Brownian motion.

This decomposition formula (6) is, in fact, equivalent to the semimartingale decomposition of $(B_t; t \leqslant T)$ in the enlarged filtration $\mathscr{B}_t^{(T)} := \mathscr{B}_t \vee \sigma(B_T)$, where $\mathscr{B}_t = \sigma\{B_s; s \leqslant t\}$:

$$B_t = \gamma_t^{(T)} + \int_0^t \mathrm{d}s \, \frac{B_T - B_s}{T - s}, \tag{7}$$

where $(\gamma_t^{(T)}; t \leqslant T)$ is a $(\mathscr{B}_t^{(T)}; t \leqslant T)$-Brownian motion, which, in particular, is independent of B_T. See [11,13] for a discussion of (6) and (7). See also [5] for a parallel between Brownian bridges and gamma bridges.

(2.2). It has been shown by Jacod–Protter [12] that formula (7) in fact extends to any integrable Lévy process $(\xi_t; t \geqslant 0)$ in the following way:

$$\xi_t = M_t^{(T)} + \int_0^t \mathrm{d}s \, \frac{\xi_T - \xi_s}{T - s}, \tag{8}$$

where $(M_t^{(T)}; t \leqslant T)$ is a martingale in the enlarged filtration $\mathscr{F}_t^{(T)} = \mathscr{F}_t \vee \sigma(\xi_T)$, where $\mathscr{F}_t = \sigma(\xi_s; s \leqslant t)$.

(2.3). Here is the proof of part (ii) of Theorem 2:

($\Rightarrow$) Let H be a harness and $s < t < T$.

Define $M_t^{(T)} = H_t - \int_0^t ((H_T - H_u)/(T-u))\, \mathrm{d}u$. Then, the harness property implies

$$\begin{aligned}
&\mathbb{E}[M_t^{(T)} | \mathscr{F}_{s,T}] \\
&\quad = \mathbb{E}[H_t | \mathscr{F}_{s,T}] - \int_0^s \frac{H_T - H_u}{T-u}\, \mathrm{d}u - \int_s^t \mathbb{E}\left[\frac{H_T - H_u}{T-u} \middle| \mathscr{F}_{s,T}\right] \mathrm{d}u \\
&\quad = \frac{T-t}{T-s} H_s + \frac{t-s}{T-s} H_T - \int_0^s \frac{H_T - H_u}{T-u}\, \mathrm{d}u - \int_s^t \frac{H_T - H_s}{T-s}\, \mathrm{d}u \\
&\quad = M_s^{(T)}.
\end{aligned}$$

($\Leftarrow$) First, remark it is enough to show that, for all $s < t < T$,

$$\mathbb{E}\left[\frac{H_t - H_s}{t-s} \middle| \mathscr{F}_{s,T}\right] = \frac{H_T - H_s}{T-s}. \tag{9}$$

Indeed, assuming that (9) holds, then, if $r < s < t < T$,

$$\begin{aligned}
\mathbb{E}\left[\frac{H_t - H_s}{t-s} \middle| \mathscr{F}_{r,T}\right] &= \mathbb{E}\left[\mathbb{E}\left[\frac{H_t - H_s}{t-s} \middle| \mathscr{F}_{s,T}\right] \middle| \mathscr{F}_{r,T}\right] \\
&= \mathbb{E}\left[\frac{H_T - H_s}{T-s} \middle| \mathscr{F}_{r,T}\right] \\
&= \frac{\mathbb{E}[H_r - H_s | \mathscr{F}_{r,T}]}{T-s} + \frac{H_T - H_r}{T-s} \\
&= \frac{r-s}{T-s} \frac{H_T - H_r}{T-r} + \frac{H_T - H_r}{T-s} \\
&= \frac{H_T - H_r}{T-r}.
\end{aligned}$$

Thus, it only remains to prove formula (9). The assumed decomposition formula (4) yields to

$$H_t - H_s = M_t^{(T)} - M_s^{(T)} + \int_s^t \mathrm{d}v \frac{H_T - H_v}{T-v}.$$

Therefore,

$$\begin{aligned}
\mathbb{E}[H_t - H_s | \mathscr{F}_{s,T}] &= \int_s^t \mathrm{d}v \frac{\mathbb{E}[H_T - H_v | \mathscr{F}_{s,T}]}{T-v} \\
&= \int_s^t \frac{\mathrm{d}v}{T-v} (H_T - H_s) - \int_s^t \frac{\mathrm{d}v}{T-v} \mathbb{E}[H_v - H_s | \mathscr{F}_{s,T}].
\end{aligned}$$

Hence, s and T being fixed, $\phi(t) := \mathbb{E}[H_t - H_s | \mathscr{H}_{s,T}]$ solves the following first-order linear differential equation:

$$\phi(t) = \int_s^t \frac{\mathrm{d}v}{T-v}(H_T - H_s) - \int_s^t \frac{\mathrm{d}v}{T-v}\phi(v); \quad s \leqslant t \leqslant T.$$

But this equation admits only one solution vanishing at s and a standard computation yields to $\phi(t) = \frac{H_T - H_s}{T-s}(t-s)$, which is formula (9).

Remark 3. Contrary to the very definition of a harness, this proposition exhibits a privileged direction of time. So a similar representation property with the opposite time direction can be derived. Namely, a measurable process H is a harness on $[0, T]$, if and only if, for all $T > \tau > 0$, there exists $(N_t^{(\tau)}; \tau < t \leqslant T)$ a $(\mathscr{F}_{\tau,t};\ \tau < t \leqslant T)$-reverse martingale such that

$$\forall \tau < t \leqslant T, \quad H_t = N_t^{(\tau)} - \int_t^T \mathrm{d}s\, \frac{H_\tau - H_s}{\tau - s}. \tag{10}$$

3. A Girsanov proof of the decomposition formula

(3.1). It is well known (see e.g. [8]) that the law of the bridge of a "good" Markov process is locally equivalent to the law of this Markov process; more precisely, if X is a Markov process taking values in $\mathbb{R}$, with $p_t(x,y)$ as its semigroup density from x to y, then the following absolute continuity relationship between $\mathbb{P}^t_{x\to y}$, the law of the bridge of length t from x to y, and $\mathbb{P}_x$ the law of X starting at x holds:

$$\mathbb{P}^t_{x\to y|\mathscr{F}_s} = \frac{p_{t-s}(X_s, y)}{p_t(x,y)}.\mathbb{P}_{x|\mathscr{F}_s}. \tag{11}$$

If $X := \xi$ is a Lévy process, $\phi_t(.)$ will denote the density of the law of ξ_t, assuming it exists and it is differentiable (see [21, p. 190] for conditions on a Lévy process to have such a density). Equality (11) then becomes

$$\mathbb{P}^t_{x\to y|\mathscr{F}_s} = \frac{\phi_{t-s}(y-\xi_s)}{\phi_t(y-x)}.\mathbb{P}_{x|\mathscr{F}_s}. \tag{12}$$

We now work in the context of such a Lévy process $(\xi_t,\ t \geqslant 0)$.

Lemma 4. *If $(M_t^y; t \leqslant T, y \in \mathbb{R})$ denotes a family of random variables such that*

- *for any $y \in \mathbb{R}$, $(M_t^y; t \leqslant T)$ is a $P^T_{x\to y}$-martingale,*
- *$(t,y) \mapsto M_t^y$ is measurable,*
- *$E[|M_t^{\xi_T}|] < \infty$.*

Then $(M_t^{\xi_T}; t \leqslant T)$ remains a P_x-martingale with respect to the filtration initially enlarged with ξ_T.

Proof. Let $(M_t^y; t \leqslant T, y \in \mathbb{R})$ be such a family of $P_{x\to y}^T$-martingales; then, for all $s < t < T$ and $\Gamma_s \in \sigma(\xi_u; u \leqslant s)$,

$$\mathbb{E}_{x\to y}^T[1_{\Gamma_s}(M_t^y - M_s^y)] = 0.$$

This implies, for any bounded Borel function f,

$$\int \mathbb{P}_x(\xi_T \in \mathrm{d}y) f(y) \mathbb{E}_{x\to y}^T[1_{\Gamma_s}(M_t^y - M_s^y)] = 0.$$

Therefore,

$$\mathbb{E}_x[f(\xi_T)1_{\Gamma_s}(M_t^{\xi_T} - M_s^{\xi_T})] = 0.$$

So, $M_t^{\xi_T}$ is a P_x-martingale with respect to the filtration enlarged with ξ_T. $\square$

(3.2). If we suppose, moreover, that $\mathbb{E}[\xi_1] = 0$, then ξ is a P_x-martingale (in any other case, we will study the Lévy process $\xi_t - \mathtt{d}t$ where $\mathtt{d}$ is the drift term of ξ). We shall denote by (σ^2, v) its local characteristics (Brownian coefficient and Lévy measure) and by $\mathscr{L}$ its infinitesimal generator. Note that $\tilde{\mathscr{L}}$, the infinitesimal generator of the time–space process (t, ξ_t) satisfies

$$\tilde{\mathscr{L}} = \frac{\partial}{\partial t} + \mathscr{L}.$$

Because of Girsanov theorem and the absolute continuity relationship (12), the process

$$\xi_t - \int_0^t \frac{\mathrm{d}\langle \xi_\cdot, \phi_{T-\cdot}(y - \xi_\cdot)\rangle_s}{\phi_{T-s}\,(y - \xi_s)}$$

defines a $P_{x\to y}^T$-martingale and therefore

$$\xi_t - \int_0^t \frac{\mathrm{d}\langle \xi_\cdot, \phi_{T-\cdot}(\xi_T - \xi_\cdot)\rangle_s}{\phi_{T-s}\,(\xi_T - \xi_s)}$$

is a P_x-martingale with respect to the filtration enlarged with ξ_T; this process will now be compared with $(M_t^{(T)})_{t\leqslant T}$ in part (ii) of Theorem 2. Namely, we aim to prove that

$$\langle \xi_\cdot, \phi_{T-\cdot}\,(y - \xi_\cdot)\rangle_t = \int_0^t \frac{y - \xi_s}{T - s}\;\phi_{T-s}\,(y - \xi_s)\,\mathrm{d}s, \tag{13}$$

that is, with our notation

$$\tilde{\mathscr{L}}(x\phi_{T-s}(y - x))(s, x) = \frac{y - x}{T - s}\,\phi_{T-s}(y - x).$$

Now,

$$\tilde{\mathscr{L}}(x\phi_{T-s}(y - x))(s, x) = -\sigma^2\phi'_{T-s}(y - x) + \int v(\mathrm{d}z)\,z\phi_{T-s}(y - x - z).$$

[This computation is quite easy once we note that $(t, x) \mapsto \phi_{T-t}(y - x)$ is a space–time harmonic function.]

The following lemma concludes the proof:

Lemma 5. *For any integrable Lévy process with local characteristics* (σ^2, v) *and transition probability density* ϕ,

$$-\sigma^2 \phi'_u(x) + \int v(\mathrm{d}z)\, z\phi_u(x-z) = \frac{x}{u}\, \phi_u(x). \tag{14}$$

Proof. From the very definition of the Lévy exponent, we have

$$\int \mathrm{e}^{\mathrm{i}\lambda x}\phi_u(x)\,\mathrm{d}x = \mathbb{E}[\mathrm{e}^{\mathrm{i}\lambda\xi_u}] = \mathrm{e}^{-u\Phi(\lambda)}. \tag{15}$$

Differentiation in λ within this equality yields to

$$\mathrm{i}\int x\phi_u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = -u\Phi'(\lambda)\mathrm{e}^{-u\Phi(\lambda)}.$$

with $\Phi'(\lambda) = \sigma^2\lambda - \mathrm{i}\int v(\mathrm{d}z) z\mathrm{e}^{\mathrm{i}\lambda z}$

Replacing $\mathrm{e}^{-u\Phi(\lambda)}$ by the left-hand side of (15) and noting that

$$\lambda\int \phi_u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = \mathrm{i}\int \phi'_u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x,$$

$$\int v(\mathrm{d}z) z\mathrm{e}^{\mathrm{i}\lambda z}\int \phi_u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = \int \mathrm{d}x\,\mathrm{e}^{\mathrm{i}\lambda x}\int v(\mathrm{d}z)\, z\phi_u(x-z),$$

we obtain

$$\mathrm{i}\int x\phi_u(x)\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = -u\int \mathrm{d}x\left(-\sigma^2\phi'_u(x) + \int v(\mathrm{d}z)\, z\phi_u(x-z)\right)\mathrm{e}^{\mathrm{i}\lambda x} \qquad \square$$

Remark 6. The right side of (14) can also be interpreted, for skip-free Lévy processes, as the density of the first hitting time, thanks to Kendall's identity (see e.g. [1]).

4. A wider class of processes: the past–future martingales

(4.1). If $\mathbb{F}$ denotes a past–future filtration, the following definition generalizes the notion of an $\mathbb{F}$-harness:

Definition 7. The two-parameter process $(M_{s,t})_{0\leqslant s<t<\infty}$ is said to be a past–future martingale with respect to $\mathbb{F} = (\mathscr{F}_{s,t})_{0\leqslant s<t<\infty}$ if:

1. $\forall s<t$, $\mathbb{E}[|M_{s,t}|] < \infty$,
2. $\forall s<t$, $M_{s,t}$ is $\mathscr{F}_{s,t}$-measurable,
3. $\forall r<s<t<u$, $\mathbb{E}[M_{s,t}|\mathscr{F}_{r,u}] = M_{r,u}$.

Remark 8.

- As previously mentioned, a process H is an $\mathbb{F}$-harness if and only if $((H_t - H_s)/(t-s))_{0\leqslant s<t<\infty}$ is a past–future martingale.

- Note that past–future martingales are reverse martingales indexed by the intervals of $\mathbb{R}^+$.

(4.2). Here we present some nontrivial past–future martingales related to a standard Brownian motion $(B_t; t \geqslant 0)$.

1. Let f_+ and f_- be two square-integrable and integrable functions on $\mathbb{R}^+$ and $C \in \mathbb{R}$. Then the process $(M_{s,t})_{0 \leqslant s < t < \infty}$ defined for all $s < t$ by
$$M_{s,t} = \int_0^s f_-(u)\,\mathrm{d}B_u + \int_t^\infty f_+(u)\,\mathrm{d}B_u + \frac{B_t - B_s}{t-s}\left(C - \int_0^s f_-(u)\,\mathrm{d}u - \int_t^\infty f_+(u)\,\mathrm{d}u\right)$$
is a past–future Brownian martingale.
One notices that the stochastic integral terms associated to the functions $f_\pm$ have to be "compensated" with a harness-type term.
2. An exponential example can easily be derived from the above example. Within the same framework, the two-parameter process $(N_{s,t})_{0 \leqslant s < t < \infty}$ defined for all $s < t$ by
$$\ln N_{s,t} = M_{s,t} + \frac{1}{2}\int_0^s f_-^2(u)\,\mathrm{d}u + \frac{1}{2}\int_t^\infty f_+^2(u)\,\mathrm{d}u + \frac{t-s}{2}\left(C - \int_0^s f_-(u)\,\mathrm{d}u - \int_t^\infty f_+(u)\,\mathrm{d}u\right)^2$$
is a past–future martingale.

(4.3). The previous examples can be partly extended to more general Lévy processes.

Proposition 9. *Let ξ be an integrable Lévy process and f a C^1 function with compact support. Then, for all $s < t$,*
$$M_{s,t} := \int_0^s f(u)\,\mathrm{d}\xi_u + \int_t^\infty f(u)\,\mathrm{d}\xi_u + \frac{\xi_t - \xi_s}{t-s}\int_s^t f(u)\,\mathrm{d}u$$
defines a past–future martingale.

Proof. Indeed, thanks to the integration by parts formula
$$\begin{aligned}\mathbb{E}\left[\int_s^t f(u)\,\mathrm{d}\xi_u | \xi_t, \xi_s\right] &= f(t)\xi_t - f(s)\xi_s - \int_s^t \mathbb{E}[\xi_u|\xi_t,\xi_s]\,\mathrm{d}f(u)\\ &= \xi_t\left(f(t) - \int_s^t \frac{u-s}{t-s}\,\mathrm{d}f(u)\right) + \xi_s\left(-f(s) - \int_s^t \frac{t-u}{t-s}\,\mathrm{d}f(u)\right)\\ &= \frac{(\xi_t - \xi_s)}{t-s}\int_s^t f(u)\,\mathrm{d}u.\end{aligned}$$

Therefore,

$$M_{s,t} = \mathbb{E}\left[\int_0^{\infty} f(u)\, \mathrm{d}\xi_u | \mathscr{H}_{s,t} \right]. \qquad \square$$

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